$w = +1\cdot40$ is here but one of the values; for, assuming first $w' = +11\cdot0$, we find $w'' = +11\cdot6$, $w''' = +11\cdot9$, and $w^{(4)} = w^{(5)} = +12\cdot0$ with the weights: $p_1 = 0\cdot170$, $p_2 = 0\cdot007$, $p_4 = 0\cdot063$. A third value of w will be found $= +9\cdot0$, and the weights $p_1 = 0\cdot047$, $p_2 = 0\cdot013$, $p_3 = 0\cdot014$, $p_4 = 0\cdot012$. The problem is indeterminate.

Excluding the first value of $\frac{[v]}{n}$ we obtain $w = +1\cdot38$, and no other value satisfying the conditions

$$w = \frac{p_2[v_2] + p_3[v_3] + p_4[v_4]}{n_2 p_2 + n_3 p_3 + n_4 p_4},$$

the weights are now:

$$p_2 = 3\cdot7,\ p_3 = 0\cdot32,\ p_4 = 0\cdot34.$$

The problem is determinate.

Washington, 1874, *May*.

Remark on the Influence of Errors of Observation on the Determination of the Orbit of a Planet from three Observations. By Herr F. W. Berg.

(*Translation.*)

Let a be the semiaxis major of the orbit, e ($= \sin\phi$) the excentricity, c the mean anomaly of the epoch, ϖ the distance of perihelion from ascending node, Ω the longitude of ascending node, i the inclination to the ecliptic; and moreover let α and β be the longitude and latitude for the first observation, α' and β' for the second, and α'' and β'' for the third. Then, as is known, compare Gauss, *Theoria Motus*, Art. 76–76:

$$\begin{aligned} d\alpha &= \mathrm{A}\,da + \mathrm{E}\,de + \mathrm{C}\,dc + \mathrm{W}\,d\varpi + \mathrm{I}\,di + \mathrm{O}\,d\Omega, \\ d\beta &= \mathrm{a}\,da + \mathrm{e}\,de + \mathrm{c}\,dc + \mathrm{w}\,d\varpi + \mathrm{i}\,di + \mathrm{o}\,d\Omega, \end{aligned} \tag{1}$$

and similarly for the other two observations.

The coefficients A, E, &c., depend on the time t and the elements a, e, &c., of the orbit, and we have thus in these equations the relation between the variations of α and β and those of the elements. From the equations (1) and the corresponding equations for the other two observations can be obtained new equations serving to express the variations of the elements in terms of the variations $d\alpha$, $d\beta$, $d\alpha'$, $d\beta'$, $d\alpha''$, $d\beta''$. We obtain

$$\begin{aligned} \Delta\,da &= (\alpha,a)\,d\alpha + (\alpha',a)\,d\alpha' + (\alpha'',a)\,d\alpha'' + (\beta,a)\,d\beta + (\beta',a)\,d\beta' + (\beta'',a)\,d\beta'', \\ \Delta\,de &= (\alpha,e)\,d\alpha + (\alpha',e)\,d\alpha' + (\alpha'',e)\,d\alpha'' + (\beta,e)\,d\beta + (\beta',e)\,d\beta' + (\beta'',e)\,d\beta'. \end{aligned} \tag{2}$$

&c.

Considering now the quantities $d\alpha$, $d\beta$, &c., as errors of

observation, these it is clear will exercise the greatest influence on the values of the elements when the determinant Δ vanishes. To investigate when this happens: the value of the determinant is

$$\Delta = \begin{vmatrix} \mathrm{A\,a} & \mathrm{A'\,a'} & \mathrm{A''\,a''} \\ \mathrm{E\,e} & \mathrm{E'\,e'} & \mathrm{E''\,e''} \\ \mathrm{C\,c} & \mathrm{C'\,c'} & \mathrm{C''\,c''} \\ \mathrm{W\,w} & \mathrm{W'w'} & \mathrm{W''w''} \\ \mathrm{I\,i} & \mathrm{I'\,i'} & \mathrm{I''\,i''} \\ \mathrm{O\,o} & \mathrm{O'\,o'} & \mathrm{O''\,o''} \end{vmatrix}. \tag{3}$$

Let x, y, z be, for the first observation, the rectangular heliocentric co-ordinates of the planet; X, Y, Z those of the Earth; and ρ the distance of the Earth and planet; then

$$\begin{aligned} x &= \rho \cos\beta \cos\alpha + \mathrm{X}, \\ y &= \rho \cos\beta \sin\alpha + \mathrm{Y}, \\ z &= \rho \sin\beta \qquad\quad + \mathrm{Z}, \end{aligned} \tag{4}$$

and similarly for the second and third observations.

Then X, Y, Z being regarded as constant; we have (compare Gauss, *Theoria Motus*, Art. 76):

$$\begin{aligned} d\alpha &= 0\,dz + \frac{\cos\alpha}{\rho\cos\beta}\,dy - \frac{\sin\alpha}{\rho\cos\beta}\,dx, \\ d\beta &= \frac{\cos\beta}{\rho}\,dz - \frac{\sin\alpha\sin\beta}{\rho}\,dy - \frac{\sin\beta\cos\alpha}{\rho}\,dx. \end{aligned} \tag{5}$$

If now v be the true anomaly of the planet, and r its radius vector (both for the first observation) the co-ordinates x, y, z can also be expressed as follows:

$$\begin{aligned} x &= r\{\cos(v+\varpi)\cos\Omega - \sin(v+\varpi)\sin\Omega\cos i\}, \\ y &= r\{\cos(v+\varpi)\sin\Omega + \cos(v+\varpi)\cos\Omega\cos i\}, \\ z &= r.\sin(v+\varpi)\sin i; \end{aligned} \tag{6}$$

and from these at once

$$\begin{aligned} \frac{dx}{d\Omega} &= -y, & \frac{dx}{di} &= z\sin\Omega, \\ \frac{dy}{d\Omega} &= x, & \frac{dy}{di} &= -z\cos\Omega, \\ \frac{dz}{d\Omega} &= 0, & \frac{dz}{di} &= z\cos z; \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{dx}{d\varpi} &= -y\cos i - x\cos\Omega\sin i, \\ \frac{dy}{d\varpi} &= x\cos i - z\sin\Omega\sin i, \\ \frac{dz}{d\varpi} &= (x\cos\Omega + y\sin\Omega)\sin i; \end{aligned} \tag{8}$$

$$\begin{aligned}
\frac{dx}{dc} &= \frac{a}{\cos\phi}\left\{\frac{p}{r^2}\frac{dx}{d\varpi} + e\sin v\,\frac{x}{r}\right\}, \\
\frac{dx}{da} &= \frac{x}{a} - \frac{3mt}{2a}\frac{dx}{dc}. \\
\frac{dx}{de} &= -a\,\frac{x}{r}\cos v + \frac{p+r}{\cos^2\phi}\,\frac{\sin v}{r}\,\frac{dx}{d\varpi},
\end{aligned} \qquad (9)$$

where m is the mean daily motion of the planet, and t is the time; also $p = a\cos^2\phi, = a\,(1-e^2)$. In the last three equations (9), x may be changed into y, or into z. In the equations (6), (7), (8), and (9), we may for x, y, z, write x', y', z', or x'', y'', z'', changing r, v, t into $r', v', t', r'', v'', t''$ accordingly.

From these partial variations of the co-ordinates in respect to the elements, the total variations may be deduced; we have

$$dx = \frac{dx}{da}da + \frac{dx}{de}de + \frac{dx}{dc}dc + \frac{dx}{d\varpi}d\varpi + \frac{dx}{di}di + \frac{dx}{d\Omega}d\Omega. \qquad (10)$$

&c.

Introducing this and the like expressions for $y, z, x', y', z', x'', y'', z''$, into the expressions (5), we obtain the variations $d\alpha$, $d\beta$, $d\alpha'$, &c., expressed in the terms of the variations da, de, dc, &c., as the equations (1) require.

The formation of the values of the coefficients A, a, &c., is also easy. We have, for example :

$$\begin{aligned}
\mathrm{A} &= \frac{1}{\rho}\left\{\frac{\cos\alpha}{\cos\beta}\frac{dy}{da} - \frac{\sin\alpha}{\cos\beta}\frac{dx}{da}\right\}, \\
\mathrm{A}' &= \frac{1}{\rho'}\left\{\frac{\cos\alpha'}{\cos\beta'}\frac{dy'}{da} - \frac{\sin\alpha'}{\cos\beta'}\frac{dx'}{da}\right\}.
\end{aligned} \qquad (11)$$

&c.

In order to make the determinants more manageable, the terms may be collected as follows; write

$$\begin{vmatrix} \mathrm{A}, & \mathrm{a} \\ \mathrm{E}, & \mathrm{e} \end{vmatrix} = \mathrm{P}, \quad \begin{vmatrix} \mathrm{C}, & \mathrm{c} \\ \mathrm{W}, & \mathrm{w} \end{vmatrix} = \mathrm{Q}, \quad \begin{vmatrix} \mathrm{I}, & \mathrm{i} \\ \mathrm{O}, & \mathrm{o} \end{vmatrix} = \mathrm{T}; \qquad (12)$$

and similarly for P′, P″, &c.; viz. P = Ae − Ea, &c.; then the determinant may be written :

$$\Delta = \begin{vmatrix} \mathrm{P}, & \mathrm{P}', & \mathrm{P}'' \\ \mathrm{Q}, & \mathrm{Q}', & \mathrm{Q}'' \\ \mathrm{T}. & \mathrm{T}', & \mathrm{T}'' \end{vmatrix}. \qquad (13)$$

We hence obtain

$$\Delta = \mathrm{L}\begin{vmatrix} \mathrm{M}\frac{\mathrm{R}}{\rho}\sin(\mathrm{A}-\Omega), & \mathrm{M}'\frac{\mathrm{R}'}{\rho'}\sin(\mathrm{A}'-\Omega), & \mathrm{M}''\frac{\mathrm{R}''}{\rho''}\sin(\mathrm{A}''-\Omega) \\ \mathrm{N}\frac{\mathrm{R}}{\rho}\sin(\mathrm{A}-\Omega), & \mathrm{N}'\frac{\mathrm{R}'}{\rho'}\sin(\mathrm{A}'-\Omega), & \mathrm{N}''\frac{\mathrm{R}''}{\rho''}\sin(\mathrm{A}''-\Omega) \\ \frac{\sin u\cos\psi}{\cos\beta}, & \frac{\sin u'\cos\psi'}{\cos\beta'}, & \frac{\sin u''\cos\psi''}{\cos\beta''}; \end{vmatrix} \qquad (14)$$

where

$$
\begin{aligned}
\mathrm{M} &= \frac{p+r}{p\,r} \sin v - \frac{3\,a\,m\,t}{2\,r^2 \cos\psi} \,(e+\cos\epsilon) \\
\mathrm{M}' &= \frac{p+r'}{p\,r'} \sin v' - \frac{3\,a\,m\,t'}{2\,r'^2 \cos\psi'} \,(e+\cos\epsilon') \\
\mathrm{M}'' &= \frac{p+r''}{p\,r''} \sin v'' - \frac{3\,a\,m\,t''}{2\,r''^2 \cos\psi''} \,(e+\cos\epsilon'') \qquad (15) \\
\mathrm{N} &= \frac{a\,e \sin v}{r \cos\psi}, \quad \mathrm{N}' = \frac{a\,e \sin v'}{r' \cos\psi'}, \quad \mathrm{N}'' = \frac{a\,e \sin v''}{r'' \cos\psi''} \\
\mathrm{L} &= -\left(\frac{r\,r'\,r''}{\rho\,\rho'\,\rho''}\right)^2 \sin^2 i \cos i\,;
\end{aligned}
$$

and ε, ε', ε'' are the excentric anomalies, and u, u', u'' the arguments of latitude of the planet, R, R$'$, R$''$, the radius vectors of the Earth, A, A$'$, A$''$ its longitudes, and ψ, ψ', ψ'' the angles at the planet (in the triangle Sun, Planet, Earth), which, as is known, Gauss first introduced into the problem.

From the expression (14) it appears that the determinant Δ vanishes when

(1.) The planet in one of the three observations is at once in the ecliptic, and in opposition with the Sun, for then the corresponding A is $=\Omega$, and the corresponding $u=0$.

(2.) $i=0$, or $=\frac{\pi}{2}$.

(3.) $\psi=\psi'=\psi''=\frac{\pi}{2}$: in such a case the passage from ρ to r is very uncertain.

(4.) $\mathrm{M\,N'-M'\,N = M'\,N''-M''\,N' = M''\,N - M\,N''} = 0$: one of these three equations being always a consequence of the other two. From the conditions of this case relations may be obtained between the elements of the orbit, and the intervals θ, θ'', which may be introduced $t=t'-\theta$, $t''=t'+\theta''$.

(5.) $e=0$; viz., when the orbit is circular.

If the three positions of the planet and those of the Sun lie in one and the same great circle, the determinant is then very small; in such a case the difference between the longitudes of the Sun's positions and that of the node Ω is also very small, and the planet is then usually in opposition.

Observatory, Wilna, 1874, *May* 15/3.